Theoretical Mechanics
Final Exam
December 11, 2018

1. (20 Pts.) Consider three pendula, each of length l, that are coupled by identical springs with spring constant k. The pendula are horizontally separated by the natural rest length of the springs d. The outer pendula have a mass m and the center pendulum has a mass $2m$.
   a. Draw a suitable diagram for this problem, letting ϕ_i represent the angle of the pendula.
   b. Assuming the angles ϕ_i remain small, show that the Lagrangian of this system is

$$L=\frac{1}{2}ml^2\dot{\phi}_1^2+ml^2\dot{\phi}_2^2+\frac{1}{2}ml^2\dot{\phi}_3^2$$
$$-\left(\frac{1}{2}mgl\phi_1^2+mgl\phi_2^2+\frac{1}{2}mgl\phi_3^2+\frac{1}{2}kl^2\left(\phi_2-\phi_1\right)^2+\frac{1}{2}kl^2\left(\phi_3-\phi_2\right)^2\right)$$

   c. Evaluate the mass and potential matrices and write the eigenvalue equation (do not attempt to solve).

2. (20 Pts.) Consider a canonical transformation generated by

$$S_2\left(q^1,\cdots,q^n,P_1,\cdots,P_n\right)=\sum_{i=1}^{n}q^iP_i+\varepsilon G\left(q^1,\cdots,q^n,P_1,\cdots,P_n\right)$$

   where ε is an infinitesimal quantity.
   a. By neglecting any order ε^2 or higher terms, show that the resulting canonical transformation differs from the identity transformation by terms of order ε with

$$P_i=p_i-\varepsilon\frac{\partial G}{\partial q^i}$$
$$Q^i=q^i+\varepsilon\frac{\partial G}{\partial P_i}=q^i+\varepsilon\frac{\partial G}{\partial p_i}$$

   Specifically, why is the second equality in the Q^i equation valid?
   b. Under this canonical transformation, show that the function $F\left(q^1,\cdots,q^n,p_1,\cdots,p_n\right)$ changes by an amount $dF\equiv F\left(Q^1,\cdots,Q^n,P_1,\cdots,P_n\right)-F\left(q^1,\cdots,q^n,p_1,\cdots,p_n\right)=\varepsilon\left[F,G\right]$ to linear order in ε where $\left[F,G\right]$ is the Poisson Bracket.
   c. If G is a constant of the motion of the Hamiltonian flow with Hamiltonian H, what is dH? What can you conclude? (Hint: Converse of Noether's Theorem)

3. (25 Pts.) Consider a string of uniform mass density σ with fixed end points and initial configuration

$$u\left(x=0,t\right)=0=u\left(x=L,t\right)$$
$$u\left(x,t=0\right)=f\left(x\right)=a\sin\left(\frac{3\pi}{L}x\right)$$
$$\frac{\partial u}{\partial t}\left(x,t=0\right)=0$$

   a. Write down the Lagrangian of this system assuming a uniform tension τ in the string. Then use the Euler-Lagrange equation to derive the equation of motion for the string.

b. Introduce a linear damping force on the string. This change will modify the equation of motion to,

$$\sigma \frac{\partial^2 u}{\partial t^2} = \tau \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}$$

Explain why β must be a positive quantity.

c. Substitute a solution of the form $u(x,t) = \rho(x)\phi(t)$ and into the equation of motion in part b. Use separation of variables then the boundary and initial conditions to determine the eigenfunctions $\rho_n(x)$, and the space mode of the solution (don't solve for $\phi(t)$ yet).

d. Show that $\phi(t)$ will have a functional form of $\phi(t) \propto e^{Ft}\left[\cos(Gt) - (F/G)\sin(Gt)\right]$, if

$$\beta^2 < \frac{36\pi^2}{L^2}\sigma\tau.$$

You need not determine the coefficients *F*,*G*. [Hint: After separating variables in part c., assume $\phi(t) = e^{\gamma t}$ where γ is a constant.]

4. (20 Pts.) We have discussed in class the solution of the wave equation for two point sources, located at $z = \pm d$, Problem 9.14. In the specific case that the sources are in phase, the far field radiated power (solid) angular distribution is

$$\frac{dP}{d\Omega} = \frac{P_0}{2\pi}\left[1 + \cos\left((4\pi d/\lambda)\cos\theta\right)\right]$$

where P_0 is the power radiated by a single source, λ is the radiation wavelength, and θ is the usual polar angle with $\theta = 0$ along the z axis.

a. Assume $\lambda = 4d$. This means there is one half wavelength change in the wave from one point source to the other. Calculate the locations θ that are maxima or minima in the power per unit solid angle.

b. What are the values of the angular power at the maxima and minima? Explain physically.

c. Now assume $\lambda = d$. Calculate locations θ of angular power maxima and minima. How many maxima and minima are there? Explain. [Hint: $\cos\theta$ varies between 1 and -1 as θ varies between 0 and π .]

d. Suppose one has a single point source and a reflecting wall. How should one arrange the source to get the same wave field for $z > 0$ as in Problem 9.14?

5. (15 Pts.) In understanding both the wave equation and heat equation, the eigenfunctions of the three dimensional Helmholtz equation

$$\nabla^2\Phi + k^2\Phi = 0$$

are important.

a. Show the functions $\Phi_{\alpha,\beta,\gamma}(x,y,z) = e^{i\alpha x}e^{i\beta y}e^{i\gamma z}$ are eigenfunctions and compute the eigenvalue k in terms of α , β , and γ .

b. What are the purely real eigenfunctions and associated eigenvalues whose values vanish at values $x = 0, a,\ \ y = 0, b,$ and $z = 0, c$? What is the frequency of the lowest non-zero mode of a cube having $b = c = a$?

c. What are the purely real eigenfunctions and associated eigenvalues whose derivatives vanish at values $x = 0, a,\ \ y = 0, b,$ and $y = 0, c$?